— Exercises —

1. **Radius of convergence.** Compute the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ in the two following cases:

   (a) $a_n = \frac{P(n)}{Q(n)}$, where P and Q are two polynomial functions. *Hint: if* $\lim_{n\to\infty} |a_{n+1}/a_n| = l$ *then* $R = 1/l$.

   (b) a_n is the n-th digit in the decimal expansion of π. *Hint: a_n is bounded and π is not a decimal number.*

2. **Manipulating power series.**

   (a) Prove that $\frac{ln(1-x)}{1-x}$ expands as a power series at $x_0 = 0$ as $-\sum_{k\geq 1}\left(\sum_{j=1}^{k}\frac{1}{j}\right)x^k$.

   (b) Find a power series for $F(x) = \int_0^x \frac{ln(1-t)}{t}dt$ at $x_0 = 0$. For which x does this series converges?

— Problems —

3. **Analytic functions.**
   Definition: Let $I \subset \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is called *real analytic* when

$$\forall x_0 \in I, \exists R > 0 \text{ s.t. } \forall x \in (x_0 - R, x_0 + R), \quad f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

   for some coefficients $(a_n)_{n\geq 0}$ depending on x_0.
   Note: the above series is called a power series centered at x_0.

   (a) Compare the set $\mathcal{C}^{\omega}(\mathbb{R})$ of analytic functions on $\mathbb{R}$ to the set $\mathcal{C}^{\infty}(\mathbb{R})$.

   (b) Show the analytic continuation property: if two analytic functions f and g on I coincide on an open subset $U \subset I$, then they coincide everywhere on I.

   (c)* Show that a power series converging on an interval I defines an analytic function.

   (d) We put $f(x) = \frac{1}{1+x^2}$ for every $x \in \mathbb{R}$. We want to show that f is analytic at 1 (and at -1) but that the power series of f at 0 has radius only 1.

      i. Determine the power series expansion of f at 0 and compute its radius.
      ii. Check that $(2 + 2x + x^2)\sum_{n\geq 0} a_n x^n = 1$ implies $a_0 = \frac{1}{2}$, $a_1 = \frac{-1}{2}$ and $a_{n+2} = -a_{n+1} - \frac{a_n}{2}$ for every $n \geq 0$.
      iii. Show that there exist two real numbers M and C such that $|a_n| \leq M \cdot C^n$ for every $n \geq 0$. Deduce from this that the series $\sum_{n\geq 0} a_n x^n$ has a nonzero radius.
      iv. Deduce from this that f is analytic at 1.

4. **Solving differential equations using power series.**
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